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Harmonic univalent functions related to q–derivative based on basic hypergeometric function

Abstract. We study a family of harmonic univalent functions using an operator involving q–derivative and hypergeometric function. Precisely we obtain a necessary and sufficient condition for functions in this family. Extreme points and convexity of such functions are also introduced.

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1 Introduction

Let $\mathcal{S}_{\mathcal{H}}$ denote the class of functions which are harmonic, univalent, complex valued and sense preserving in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in \mathcal{S}_{\mathcal{H}}$ is of the type $f = h + \overline{g}$ where h and g are analytic in $\mathbb{U}$. We call h and g analytic part and co-analytic part of f respectively. Also f is locally univalent and sense preserving in $\mathbb{U}$ if and only if $|h'(z)| > |g'(z)|$ in $\mathbb{U}$, see [2]. Thus, for $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, we may consider:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \qquad (0 \leqslant b_1 < 1). \tag{1}$$

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The q–shifted factorial for $|q|<1$ defined by:

$$(\alpha,q)_k=\begin{cases}1 & , \quad k=0,\\ (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{k-1}) & , \quad k\in\mathbb{N},\end{cases} \tag{2}$$

where $\mathbb{N}$ denote the set of positive integers and α is a complex number.

For complex parameters α_i, β_j and q where $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, $\beta_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}$ and $|q|<1$, we consider the basic hypergeometric function ${}_m\Phi_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n,q,z)$ by:

$${}_m\Phi_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n,q,z)=\sum_{k=0}^{\infty}\frac{(\alpha_1,q)_k\cdots(\alpha_m,q)_k}{(q,q)_k(\beta_1,q)_k\cdots(\beta_n,q)_k}z^k, \tag{3}$$

where $m=n+1$, $m,n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $z\in\mathbb{U}$ and the q–shifted factorial $(\alpha,q)_k$ is given in (2).

We note that

$$\begin{aligned}&\lim_{q\to1^-}\left({}_m\Phi_n\big(q^{\alpha_1},\ldots,q^{\alpha_m};q^{\beta_1},\ldots,q^{\beta_n},q,(q-1)^{1+n-m}z\big)\right)\\ &={}_mF_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n,q,z),\end{aligned} \tag{4}$$

where ${}_mF_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n,q,z)$ is the well-known hypergeometric function. For more details, one may refer to [3, 5] and [6].

The q–derivative of a function G is defined by:

$$\partial_q\big(G(z)\big)=\frac{G(qz)-G(z)}{(q-1)z}, \qquad (q\neq1, \quad z\neq0). \tag{5}$$

We can easily observe that:

$$\partial_q\big(z^k\big)=\frac{1-q^k}{1-q}z^{k-1}=[k]_qz^{k-1}, \tag{6}$$

where $[k]_q=\frac{1-q^k}{1-q}$ is the q–integer number, see [7] and [8].

We conclude that:

$$\lim_{q\to1}\partial_q\big(G(z)\big)=G'(z).$$

For more properties of q–derivative, see [4] and [7]. Now, we consider the linear operator:

$$\begin{aligned}&\mathcal{H}_n^m(\alpha_1,\dots,\alpha_m;\beta_1,\dots,\beta_n;q)f(z)\\&=\big(z\ {}_m\Phi_n(\alpha_1,\dots,\alpha_m;\beta_1,\dots,\beta_n;q,z)\big)*f(z)\\&=z+\sum_{k=2}^{\infty}\Gamma(\alpha_i,\beta_j,q,k)a_kz^k,\end{aligned}\tag{7}$$

where "$*$" stands for well-known convolution (or Hadamard product) and:

$$\Gamma(\alpha_i,\beta_j,q,k)=\frac{(\alpha_1,q)_{k-1}\cdots(\alpha_m,q)_{k-1}}{(q,q)_{k-1}(\beta_1,q)_{k-1}\cdots(\beta_n,q)_{k-1}}.\tag{8}$$

It is convenient to write:

$$\mathcal{H}_n^m(\alpha_1,\dots,\alpha_m;\beta_1,\dots,\beta_n;q)f(z)=\mathcal{H}_n^m(\alpha,\beta,q).\tag{9}$$

H. Aldweby and M. Darus [1] were defined the operator (7) on harmonic function $f=h+\overline{g}$ given by (1) as follows:

$$\mathcal{H}_n^m(\alpha,\beta,q)f(z)=\mathcal{H}_n^m(\alpha,\beta,q)h(z)+\overline{\mathcal{H}_n^m(\alpha,\beta,q)g(z)}.\tag{10}$$

For more properties of operators given in (7) and (10), see [3].

We denote by $\mathcal{S}_{\overline{\mathcal{H}}}$ the class of functions $f=h+\overline{g}$, where:

$$h(z)=z-\sum_{k=2}^{\infty}|a_k|z^k,\qquad g(z)=\sum_{k=1}^{\infty}|b_k|z^k,\qquad (|b_1|<1).\tag{11}$$

For $\gamma\geqslant 0$, $0\leqslant\delta,\eta\leqslant 1$, $0\leqslant\sigma<1$ and $t\in\mathbb{R}$ let $\mathcal{S}^t_{\overline{\mathcal{H}}}(\gamma,\delta,\eta,\sigma)$ denote the class of functions in $\mathcal{S}_{\mathcal{H}}$ of the type (1) such that:

$$\begin{aligned}\mathrm{Re}\Bigg\{\big(\eta e^{it}-\gamma\delta\big)-\eta e^{it}\frac{\big(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\big)''}{z''}+(\gamma+\delta)\frac{\big(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\big)'}{z'}\\+(1-\gamma)(1-\delta)\frac{\mathcal{H}_n^m(\alpha,\beta,q)f(z)}{z}\Bigg\}\geqslant\sigma,\end{aligned}\tag{12}$$

where

$$z' = \frac{\partial}{\partial\theta}(z) = iz, \qquad z'' = \frac{\partial^2}{\partial\theta^2}(z) = -z, \tag{13}$$

and

$$\begin{aligned}\big(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\big)' &= \frac{\partial}{\partial\theta}\big(\mathcal{H}_n^m(\alpha,\beta,q)f(re^{i\theta})\big)\\ &= iz\big(\mathcal{H}_n^m(\alpha,\beta,q)h\big)' - iz\overline{\big(\mathcal{H}_n^m(\alpha,\beta,q)g\big)'},\end{aligned} \tag{14}$$

$$\begin{aligned}\big(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\big)'' &= \frac{\partial^2}{\partial\theta^2}\big(\mathcal{H}_n^m(\alpha,\beta,q)f(re^{i\theta})\big)\\ &= z\big(\mathcal{H}_n^m(\alpha,\beta,q)h\big)' - z^2\big(\mathcal{H}_n^m(\alpha,\beta,q)h\big)''\\ &- z\big(\mathcal{H}_n^m(\alpha,\beta,q)g\big)' - z^2\overline{\big(\mathcal{H}_n^m(\alpha,\beta,q)g\big)''}.\end{aligned} \tag{15}$$

Also we denote by $\mathcal{S}_{\overline{\mathcal{H}}}^t(\gamma,\delta,\eta,\sigma)$ the subclass of $\mathcal{S}_{\mathcal{H}}^t(\gamma,\delta,\eta,\sigma)$ consisting of functions $f \in \mathcal{S}_{\overline{\mathcal{H}}}$ of the type (11) which satisfy the condition (12).

2 Main results

In this section, we first give the sufficient coefficient bounds for $f(z) \in \mathcal{S}_{\mathcal{H}}^t(\gamma,\delta,\eta,\sigma)$ and then we show these sufficient conditions are also necessary for $f(z) \in \mathcal{S}_{\overline{\mathcal{H}}}^t(\gamma,\delta,\eta,\sigma)$.

Theorem 2.1. *Suppose $f = h + \overline{g}$, h and g be given by (1) and:*

$$\begin{aligned}&\sum_{k=2}^{\infty}\big|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\big|\,\Gamma(\alpha_i,\beta_j,q,k)|a_k| +\\ &\sum_{k=1}^{\infty}\big|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\big|\,\Gamma(\alpha_i,\beta_j,q,k)|b_k| \leqslant 1-\sigma.\end{aligned} \tag{16}$$

Then $f(z) \in \mathcal{S}_{\mathcal{H}}^t(\gamma,\delta,\eta,\sigma)$.

Proof. By using the fact that:

$$\mathrm{Re}\{W\} \geqslant \sigma \Longleftrightarrow |W + 1 - \sigma| \geqslant |W - 1 - \sigma|$$

and letting:

$$W = \eta e^{it} - \gamma\delta) - \eta e^{it}\frac{\left(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\right)''}{z''} + (\gamma+\delta)\frac{\left(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\right)'}{z'} + (1-\gamma)(1-\delta)\frac{\mathcal{H}_n^m(\alpha,\beta,q)f(z)}{z}$$

it is enough to show that:

$$|W+1-\sigma| - |W-1-\sigma| \geqslant 0.$$

But by using (13), (14) and (15) we have:

$$|W+1-\sigma| = \Bigg|\eta e^{it} - \gamma\delta - \eta e^{it}\Bigg(1 + \sum_{k=2}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} + \\ + \sum_{k=2}^{\infty} k(k-1)\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} + \sum_{k=1}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1} \\ + \sum_{k=1}^{\infty} k(k-1)\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\Bigg) \\ + (\gamma+\delta)\Bigg(1 + \sum_{k=2}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k} - \sum_{k=1}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\Bigg) \\ + (1-\gamma)(1-\delta)\Bigg(1 + \sum_{k=2}^{\infty} \Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} + \sum_{k=1}^{\infty} \Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\Bigg)\Bigg| \\ \geqslant 2 - \sigma - \sum_{k=2}^{\infty}\left|1 + (\gamma+\delta)(k-1) + \gamma\delta - \eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|a_k|\left|\frac{z^k}{z}\right| \\ - \sum_{k=1}^{\infty}\left|1 - (\gamma+\delta)(k-1) + \gamma\delta - \eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|b_k|\left|\frac{z^k}{z}\right|,$$

and

$$|W-1-\sigma| \leqslant \sigma + \sum_{k=2}^{\infty}\left|1 + (\gamma+\delta)(k-1) + \gamma\delta - \eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|a_k|\left|\frac{z^k}{z}\right|$$

$$+\sum_{k=1}^{\infty}\left|1-(\gamma+\delta)(k-1)+\gamma\delta-\eta k^{2}\right|\Gamma(\alpha_i,\beta_j,q,k)|b_k|\left|\frac{z^k}{z}\right|,$$

where $\Gamma(\alpha_i,\beta_j,q,k)$ is defined by (8).

So by using (16), we have:

$$\begin{aligned}&|W+1-\sigma|-|W-1-\sigma|\geqslant\\&2\Big[1-\sigma-\sum_{k=2}^{\infty}\left|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^{2}\right|\Gamma(\alpha_i,\beta_j,q,k)|a_k|\\&-\sum_{k=1}^{\infty}\left|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^{2}\right|\Gamma(\alpha_i,\beta_j,q,k)|b_k|\Big]\geqslant 0.\end{aligned}$$

□

Remark 2.2. The coefficient bound (16) is sharp for the function:

$$\begin{aligned}H(z)=z&+\sum_{k=2}^{\infty}\frac{x_k}{|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2|\,\Gamma(\alpha_i,\beta_j,q,k)}z^k\\&+\sum_{k=1}^{\infty}\frac{\overline{y_k}}{|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2|\,\Gamma(\alpha_i,\beta_j,q,k)}(\overline{z})^k,\end{aligned}$$

where

$$\frac{1}{1-\sigma}\Big(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\Big)=1.$$

Theorem 2.3. *Let $f=h+\overline{g}\in\mathcal{S}_{\overline{\mathcal{H}}}^{t}(\gamma,\delta,\eta,\sigma)$ if and only if:*

$$\begin{aligned}&\sum_{k=2}^{\infty}\Big(\left|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^{2}\right|\;|a_k|\\&+\left|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^{2}\right|\;|b_k|\Big)\Gamma(\alpha_i,\beta_j,q,k)\\&\leqslant 1-\sigma-\big(2(\gamma+\delta)-(1+\gamma\delta+\eta)\big)|b_1|.\end{aligned}\tag{17}$$

Proof. From Theorem **??** $\mathcal{S}_{\overline{\mathcal{H}}}^{t}(\gamma,\delta,\eta,\sigma)\subset\mathcal{S}_{\mathcal{H}}^{t}(\gamma,\delta,\eta,\sigma)$, and since (16) is equivalent to (17) we conclude the “if part”. For the “only if part”, suppose that $f(z) \in \mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$. Then for $z = re^{i\theta} \in \mathbb{U}$, we have:

$$
\begin{aligned}
&\operatorname{Re}\left\{ \left(\eta e^{it} - \gamma\delta\right) - \eta e^{it} \frac{\left(\mathcal{H}^m_n(\alpha,\beta,q)f(z)\right)''}{z''} + (\gamma+\delta)\frac{\left(\mathcal{H}^m_n(\alpha,\beta,q)f(z)\right)'}{z'} \right.\\
&\left. + (1-\gamma)(1-\delta)\frac{\mathcal{H}^m_n(\alpha,\beta,q)f(z)}{z} \right\} \geqslant \sigma,\\
&= \operatorname{Re}\left\{ \eta e^{it} - \gamma\delta \right.\\
&- \eta e^{it}\left(1 + \sum_{k=2}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} + \sum_{k=2}^{\infty} k(k-1)\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} \right.\\
&\left. + \sum_{k=1}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1} + \sum_{k=1}^{\infty} k(k-1)\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1} \right)\\
&+ (\gamma+\delta)\left(1 + \sum_{k=2}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} - \sum_{k=1}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1} \right)\\
&\left. + (1-\gamma)(1-\delta)\left(1 + \sum_{k=2}^{\infty} \Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} + \sum_{k=1}^{\infty} \Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1} \right) \right\}\\
&\geqslant 1 - \sum_{k=2}^{\infty} \left|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \, |a_k|\Gamma(\alpha_i,\beta_j,q,k)\\
&+ \big(2(\gamma+\delta) - (1+\gamma\delta+\eta)\big)\, |b_1|\\
&+ \sum_{k=2}^{\infty} \left|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i,\beta_j,q,k)|b_k| r^{k-1} \geqslant \sigma.
\end{aligned}
$$

The above inequality holds for all $z \in \mathbb{U}$. So if $z = r \to 1$. We obtain the required result (17). Now the proof of theorem is complete. □

3 Geometric properties

In this section we introduce extreme points of $\mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$ and show that this class is a convex set.

Theorem 3.1. $f = h + \overline{g} \in \mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$ *if and only if it can be expressed* *of the type:*

$$f(z) = X_1 z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_k(z), \qquad (z \in \mathbb{U}), \tag{18}$$

where

$$h_k(z) = z - \frac{1-\sigma}{|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2|\,\Gamma(\alpha_i, \beta_j, q, k)} z^k, \tag{19}$$

and

$$g_k(z) = \frac{1-\sigma}{|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2|\,\Gamma(\alpha_i, \beta_j, q, k)} (\overline{z})^k, \tag{20}$$

$X_1 \geqslant 0, Y_1 \geqslant 0, X_1 + \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1, X_k \geqslant 0, Y_k \geqslant 0$ *for* $k = 2, 3, \ldots,$ *and* $\Gamma(\alpha_i, \beta_j, q, k)$ *is given by* (8).

Proof. If f be given by (18), then:

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\sigma}{|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2|\,\Gamma(\alpha_i, \beta_j, q, k)} X_k z^k$$
$$+ \sum_{k=1}^{\infty} \frac{1-\sigma}{|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2|\,\Gamma(\alpha_i, \beta_j, q, k)} Y_k (\overline{z})^k.$$

Since by (17), or equivalently by (16), we have:

$$\sum_{k=2}^{\infty} \left|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k) \times$$
$$\times \frac{(1-\sigma)|X_k|}{|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2|\,\Gamma(\alpha_i, \beta_j, q, k)}$$
$$+ \sum_{k=1}^{\infty} \left|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k) \times$$
$$\times \frac{(1-\sigma)|Y_k|}{|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2|\,\Gamma(\alpha_i, \beta_j, q, k)}$$
$$= (1-\sigma)\Big(\sum_{k=2}^{\infty} |X_k| + \sum_{k=1}^{\infty} |Y_k|\Big)$$

$$= (1-\sigma)(1-X_1) \leqslant 1-\sigma.$$

So $f(z) \in \mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$.

Conversely, suppose $f(z) \in \mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$. By putting:

$$X_1 = 1 - \Big(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k\Big),$$

where

$$X_k = \frac{\big|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\big|\,\Gamma(\alpha_i,\beta_j,q,k)}{1-\sigma}|a_k|,$$
$$Y_k = \frac{\big|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\big|\,\Gamma(\alpha_i,\beta_j,q,k)}{1-\sigma}|b_k|$$

We conclude the required representation, so the proof is complete. □

Theorem 3.2. *If $f_n(z)$, $n = 1,2,\ldots$, belongs to $\mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$, then the function $F(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z)$ is also in $\mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$, where $f_n(z)$ defined by:*

$$f_n(z) = z - \sum_{k=2}^{\infty} a_{k,n} z^k + \sum_{k=1}^{\infty} b_{k,n}(\overline{z})^k, \quad \big(n=1,2,\ldots, \quad \textstyle\sum_{n=1}^{\infty}\lambda_n = 1\big). \tag{21}$$

Proof. Since $f(z) \in \mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$, by (17) or equivalently (16), for $n = 1,2,\ldots$ we have:

$$\sum_{k=2}^{\infty} \big|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\big|\,\Gamma(\alpha_i,\beta_j,q,k)|a_{k,n}|$$
$$+ \sum_{k=1}^{\infty} \big|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\big|\,\Gamma(\alpha_i,\beta_j,q,k)|b_{k,n}| \leqslant 1-\sigma.$$

Also

$$F(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z) = z - \sum_{k=2}^{\infty}\Big(\sum_{n=1}^{\infty}\lambda_n a_{k,n}\Big) z^k + \sum_{k=1}^{\infty}\Big(\sum_{n=1}^{\infty}\lambda_n b_{k,n}\Big)(\overline{z})^k,$$

Now according to (17) or equivalently (16), we have:

$$\sum_{k=2}^{\infty}\left|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\left|\sum_{n=1}^{\infty}\lambda_n a_{k,n}\right|\Gamma(\alpha_i,\beta_j,q,k)$$
$$+\sum_{k=1}^{\infty}\left|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\left|\sum_{n=1}^{\infty}\lambda_n b_{k,n}\right|\Gamma(\alpha_i,\beta_j,q,k)$$
$$=\sum_{n=1}^{\infty}\left\{\sum_{k=2}^{\infty}\left|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|a_{k,n}|\right.$$
$$\left.+\sum_{k=1}^{\infty}\left|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|b_{k,n}|\right\}\lambda_n$$
$$\geqslant(1-\sigma)\sum_{n=1}^{\infty}\lambda_n=1-\sigma.$$

Thus $F(z)\in\mathcal{S}_{\mathcal{H}}^{t}(\gamma,\delta,\eta,\sigma)$. □

Remark 3.3. We note that $\mathcal{S}_{\mathcal{H}}^{t}(\gamma,\delta,\eta,\sigma)$ is a convex set.

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